Aharonov-Casher effect for spin one particles in a noncommutative space

B. Mirza, R. Narimani and M. Zarei

Department of Physics, Isfahan University of Technology, Isfahan 84156, Iran

E-mail: b.mirza@cc.iut.ac.ir

ABSTRACT

In this work the Aharonov-Casher (AC) phase is calculated for spin one particles in a noncommutative space. The AC phase has previously been calculated from the Dirac equation in a noncommutative space using a gauge-like technique [17]. In the spin-one, we use kemmer equation to calculate the phase in a similar manner. It is shown that the holonomy receives non-trivial kinematical corrections. By comparing the new result with the already known spin 1/2 case, one may conjecture a generalized formula for the corrections to holonomy for higher spins.

PACS numbers: 02.40.Gh, 03.65.Pm.

Keywords: Noncommutative geometry; Aharonov-Casher; Topological Phases.

1 Introduction

In the last few years, theories in noncommutative space have been studied extensively (for a review see [1]). Noncommutative field theories are related to M-theory compactification [2], string theory in nontrivial backgrounds [3] and quantum Hall effect [4]. Inclusion of noncommutativity in quantum field theory can be achieved in two different ways: via Moyal $\star$-product on the space of ordinary functions, or defining the field theory on a coordinate operator space which is intrinsically noncommutative[1,5]. The equivalence between the two approaches has been nicely described in [6]. A simple insight on the role of noncommutativity in field theory can be obtained by studying the one particle sector, which prompted an interest in the study of noncommutative quantum mechanics [7,8,9,10,11,12,13,14]. In these studies some attention was paid to the Aharonov-Bohm effect [15]. If the noncommutative effects are important at very high energies, then one could posit a decoupling theorem that produces the standard quantum field theory as an effective field theory and that does not remind the noncommutative effects. However the experience from atomic and molecular physics strongly suggests that the decoupling is never complete and that the high energy effects appear in the effective action as topological remnants. Along these lines, the Aharonov-Bohm and Aharonov-Casher effects have already been investigated in a noncommutative space [16,17]. In this work, we will develop a method to obtain the corrections to the topological phase of the Aharonov-Casher effect for spin one particles, where we know that in a commutative space the line spectrum does not depend on the relativistic nature of the dipoles. The article is organized as follows; in section 2, we discuss the Aharonov-Casher effect for spin one particles on a commutative space. In section 3, the Aharonov-Casher effect in a noncommutative space is studied and a generalized formula for holonomy is given.

2 The Aharonov-Casher effect

In 1984 Aharonov and Casher (AC) [18] pointed out that the wave function of a neutral particle with nonzero magnetic moment μ develops a topological phase when traveling in a closed path which encircles an infinitely long filament carrying a uniform charge density. The AC phase has been measured experimentally [19]. This phenomenon is similar to the Aharonov-Bohm (AB) effect. The similarities and the differences of these two phenomena and possible classical interpretations of the AC effect have been discussed by several authors [20,21,22]. In Ref. [18], the topological phase of the AC effect was derived by considering a neutral particle with a nonzero magnetic dipole moment moving in an electric field produced by an infinitely long filament carrying a uniform charge density. If the particle travels over a closed path which includes the filament, a topological phase will result. This phase is given by

$$\phi_{AC} = \oint (\vec{\mu} \times \mathbf{E}) \cdot d\mathbf{r} \tag{1}$$

where $\vec{\mu} = \mu\vec{\sigma}$ is the magnetic dipole moment and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where σ_i $(i = 1, 2, 3)$ are the 2×2 Pauli matrices. It is possible to arrange that the particle moves in the x-y plane and travels over a closed path which includes an infinite filament along z-axis. The electric field in the point $\mathbf{r} = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors in the direction of the positive x and y axes, is given as

$$\mathbf{E} = \frac{\lambda}{2\pi(x^2 + y^2)}(x\hat{i} + y\hat{j}) \tag{2}$$

where λ is the linear charge density of the filament and the phase is given by

$$\phi_{AC} = \mu\sigma_3 \oint (\hat{k} \times \mathbf{E}) \cdot d\mathbf{r} = \mu\sigma_3\lambda \tag{3}$$

where $\hat{k}$ is a unit vector along z axis. This phase is purely quantum mechanical and has no classical interpretation. The appearance of σ_3 in the phase represents the spin degrees of freedom. We see that different components acquire phases with different signs. This is also one of the points that distinguishes the AC effect from the AB effect [23]. In this part, we briefly explain a method for obtaining Eq.(3). The equation of motion for a neutral spin half particle with a nonzero magnetic dipole moment moving in a static electric field $\mathbf{E}$ is given by

$$(i\gamma_\mu\partial^\mu + \frac{1}{2}\mu\sigma_{\alpha\beta}F^{\alpha\beta} - m)\psi = 0 \tag{4}$$

or it can be written as

$$(i\gamma_\mu\partial^\mu - i\mu\gamma \cdot \mathbf{E}\gamma_0 - m)\psi = 0 \tag{5}$$

where $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ and $\gamma-$matrices are defined by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad\qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{6}$$

We define

$$\psi = \ e^{\mathbf{a}f}\psi_0 \tag{7}$$

where $\mathbf{a}$ is the matrix to be determined below, f is a time independent scalar phase, and ψ_0 is a solution of the Dirac equation

$$(i\gamma_\mu\partial^\mu - m)\psi_0 = 0 \tag{8}$$

Writing ψ_0 in terms of ψ and multiplying (8) by $e^{\mathbf{a}f}$ from the left, we obtain

$$e^{\mathbf{a}f}(i\gamma^\mu\partial_\mu - m)e^{-\mathbf{a}f}\psi = 0 \tag{9}$$

$$(ie^{\mathbf{a}f}\gamma^\mu e^{-\mathbf{a}f}\partial_\mu - ie^{\mathbf{a}f}\gamma^i e^{-\mathbf{a}f}\mathbf{a}\ \partial_i f - m)\psi = 0 \tag{10}$$

Comparing Eq.(10) with Eq.(5), we find that $\mathbf{a}$ and f must satisfy

$$\mu\gamma \cdot \mathbf{E}\gamma_0 = (\gamma \cdot \nabla f)\mathbf{a} \quad , \quad \mathbf{a}\gamma_\mu = \gamma_\mu\mathbf{a} \tag{11}$$

The matrix $\mathbf{a}$ can be expressed by some linear combination of the complete set of 4×4 matrices $1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5$ and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. The second member of Eqs.(11) cannot be satisfied if all γ_1, γ_2 and γ_3 are present in Eq.(10). However, it is possible to satisfy it if the problem in question can be reduced to the planar one. This indicates that the AC topological phase can arise only in two spatial dimensions. Therefore, let us consider the particle moving in $x - y$ plane in which case only the matrices γ_1 and γ_2 are present in (11), and moreover, $\partial_3\psi$ and E_3 vanish. The choice $-i\sigma_{12}\gamma_0$ represents a consistent Ansatz. From the first equation in (11), we get

$$\nabla f = \mu(\hat{k} \times \mathbf{E}) \tag{12}$$

and the phase is given by

$$\begin{aligned} \phi^{(0)} &= \sigma_{12}\gamma_0 \oint \vec{\nabla} f \cdot d\mathbf{r} \\ &= \mu\sigma_{12}\gamma_0 \oint (\hat{k} \times \mathbf{E}) \cdot \mathbf{dr} \\ &= \mu \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \oint (\hat{k} \times \mathbf{E}) \cdot d\mathbf{r} \end{aligned} \tag{13}$$

One may extend this method to spin one using the first order kemmer equation (for a more complete explanation and derivation see [24]). The kemmer equation is defined by

$$(i\beta^\mu \partial_\mu - m)\psi = 0 \tag{14}$$

where the β-matrices are generalizations of the Dirac gamma matrices. These satisfy an algebra ring, which for spin one is

$$\beta^\mu\beta^\nu\beta^\rho + \beta^\rho\beta^\nu\beta^\mu = \eta^{\mu\nu}\beta^\rho + \eta^{\nu\rho}\beta^\mu \tag{15}$$

These Kemmer β-matrices are reducible, that is the 16×16 representation decomposes into three separate representations: a one dimensional trivial representation; a five dimensional spin zero representation; and the 10-d spin one representation [24,25,26]. This algebra is odd, that is it cannot reduce the matrix operator to the identity, unlike the Dirac algebra. In this paper we choose the 10-d spin one representation which is given by the following 10×10 matrices

$$\beta^0 = \begin{pmatrix} \widehat{O} & \widehat{O} & I & o^\dagger \\ \widehat{O} & \widehat{O} & \widehat{O} & o^\dagger \\ I & \widehat{O} & \widehat{O} & o^\dagger \\ o & o & o & 0 \end{pmatrix} \qquad \beta^j = \begin{pmatrix} \widehat{O} & \widehat{O} & \widehat{O} & -iK^{j\dagger} \\ \widehat{O} & \widehat{O} & S^j & o^\dagger \\ \widehat{O} & -S^j & \widehat{O} & o^\dagger \\ -iK^j & o & o & 0 \end{pmatrix} \tag{16}$$

where the elements are

$$\widehat{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

$$S^1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad S^2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad S^2 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{18}$$

$$o = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad K^1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad K^2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad K^3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \tag{19}$$

Just as in Dirac theory, the Lorentz invariance of the Kemmer theory entails a transformation of the spinor so that the matrix representation remain the same. The Lorentz generator for these transformations, $S_{\mu\nu}$, is proportional to the antysymmetric product of two matrices of the ring,

$$S_{\mu\nu} = b(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu) \tag{20}$$

These generators satisfy well known commutation relation and define the spin operators. The coefficient b is linked to the coefficient of the commutation relations, and is set below according to our convenience. The equation of motion for a neutral spin one particle with an anomalous magnetic moment in Kemmer theory is

$$(i\beta_\mu\partial^\mu + \frac{1}{2}\mu S_{\alpha\beta}F^{\alpha\beta} - m)\psi = 0 \tag{21}$$

The interaction term emerge from the derivation of a second order Kemmer equation following the method of Umezawa [26]. The operator component of the phase in the spin one AC phase solution is a spin one pseudo-vector operator defined by

$$\xi_\mu = \frac{i}{2}\varepsilon_{\mu\nu\lambda\rho}\beta^\nu\beta^\lambda\beta^\rho \tag{22}$$

The path dependent phase proportional to ξ_3 is introduced in the free Kemmer equation of motion [17]

$$(i\beta^\mu\partial_\mu - m)e^{i\xi_3\int^r \mathbf{A}\cdot d\mathbf{r}}\psi = 0 \tag{23}$$

with the intention to transform this into the equation of motion (21) with the anomalous magnetic moment term. Multiplying (23) by $e^{(-i\xi_3\int^r \mathbf{A}\cdot d\mathbf{r})}$ from the left and comparing with (21), we obtain

$$exp[-i\xi_3\int^r \mathbf{A}\cdot d\mathbf{r}]\ \beta^\nu exp[i\xi_3\int^r \mathbf{A}\cdot d\mathbf{r}] = \beta^\nu \tag{24}$$

$$-\beta^\nu\xi_3 A_\nu\psi = \frac{1}{2}\mu S_{\alpha\beta}F^{\alpha\beta}\psi = \mu S_{0i}F^{0i}\psi \tag{25}$$

By using the Baker-Hausdorff formula in the first condition, it is easy to see that for $\nu \neq 3$ the commutators are zero. However for $\nu = 3$ the commutators does not vanish and so the first condition restrict the dynamics to 2+1 dimensions, just as the spin half.

By a direct calculation in the second line and using definition of the ξ_3, β^ν and $S_{\mu\nu}$ matrices one has

$$A_1 = -2\mu E_2 \qquad A_2 = 2\mu E_1 \tag{26}$$

Finally the AC phase for a closed path is given by

$$\phi_{AC} = \xi_3 \oint \mathbf{A} \cdot d\mathbf{r} = 2\mu\xi_3 \int_S (\nabla \cdot E) \cdot dS = 2\mu\xi_3\lambda \tag{27}$$

3 The spin one AC effect on a noncommutative space

The noncommutative Moyal spaces can be realized as spaces where coordinate operator $\hat{x}^\mu$ satisfies the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \tag{28}$$

where $\theta^{\mu\nu}$ is an antisymmetric tensor and is of space dimension (length)^2. We note that space-time noncommutativity, $\theta^{0i} \neq 0$, may lead to some problems with unitarity and causality. Such problems do not occur for the quantum mechanics on a noncommutative space with a usual commutative time coordinate. The noncommutative models specified by Eq.(14) can be realized in terms of a $\star$-product: the commutative algebra of functions with the usual product f(x)g(x) is replaced by the $\star$-product Moyal algebra:

$$(f \star g)(x) = exp\ [\frac{i}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}]\ f(x)g(y)|_{x=y} \tag{29}$$

As for the phase space, inferred from string theory, we choose

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \tag{30}$$

The noncommutative quantum mechanics can be defined by [7-14],

$$H(p, x) \star \psi(x) = E\psi(x) \tag{31}$$

The equation of motion for a neutral spin one particle with a nonzero magnetic dipole moment moving in a static electric field $\mathbf{E}$ is given by

$$(i\beta_\mu\partial^\mu + \frac{1}{2}\mu S_{\alpha\beta}F^{\alpha\beta} - m) \star \psi = 0 \tag{32}$$

As in [17] we define

$$\psi = e^{i\xi_3 f}\psi_0 \tag{33}$$

where ξ_3 is the matrix already defined, f is a time independent scalar phase, and ψ_0 is a solution of the free Kemmer equation

$$(i\beta_\mu\partial^\mu - m)\psi_0 = 0 \tag{34}$$

By using Baker-Cample-Housdorff formula and $[\beta_\mu, \xi_3] = 0$, (32) can be written as

$$-\beta^\mu \partial_\mu(\xi_3 f) e^{i\xi_3 f}\psi_0 + \frac{\mu}{2} S_{\alpha\beta} F^{\alpha\beta} \star (e^{i\xi_3 f}\psi_0) = 0 \tag{35}$$

After expanding the second term in (35) up to the first order of the noncommutativity parameter $\theta_{ij} = \theta\epsilon_{ij}$ and defining k_j as

$$\partial_j \psi_0 = (ik_j)\psi_0 \tag{36}$$

the final result up to first order in θ is given by

$$[-\beta^\mu \partial_\mu(\xi_3 f) + \mu(S_{0l}F^{0l} + \frac{i}{2}\theta_{ij}\partial_i(S_{0l}F^{0l})\partial_j(i\xi_3 f) + \frac{i}{2}\theta_{ij}\partial_i(S_{0l}F^{0l})(ik^j))]exp[i\xi_3 f]\psi_0 = 0 \tag{37}$$

It should be noted that expansion of F^{0l} or $\mathbf{E}$ up to first order in θ leads to an additive correction to the commutative holonomy and does not cause a new non-topological behaviour. A similar situation happens in the noncommutative Aharonov-Bohm effect. By expanding f up to first order in θ

$$f = f^{(0)} + \theta f^{(1)} + ... \tag{38}$$

we obtain the following equations,

$$[-\beta^\mu \partial_\mu(\xi_3 f^{(0)}) + \mu(S_{0l}F^{0l})]\psi_0 = 0 \tag{39}$$

$$[-\beta^\mu \partial_\mu(\xi_3 f^{(1)}) + \frac{i\mu}{2}\ \varepsilon_{ij}\partial_i(S_{0l}F^{0l})\partial_j(i\xi_3 f^{(0)}) + \frac{i\mu}{2}\ \varepsilon_{ij}\partial_i(S_{0l}F^{0l})(ik^j)]\psi_0 = 0 \tag{40}$$

by choosing $b = 2$ in (20) and after a straightforward calculation we get to

$$\nabla f^{(0)} = 2\mu(\hat{k} \times \vec{E}) \tag{41}$$

which is equivalent to (26), and the phase is given by

$$\begin{aligned} \phi^{(0)} &= \xi_3 \oint \nabla f^{(0)} \cdot d\mathbf{r} \\ &= 2\mu\xi_3 \oint (\hat{k} \times \mathbf{E}) \cdot d\mathbf{r} \end{aligned} \tag{42}$$

Substituting Eq.(42) in (40) and then using the wave functions which are given in [24], and a long but straightforward calculation (the Mathematica package is used) the following correction to $\phi^{(0)}$ for a neutral particle with nonzero magnetic dipole moment μ and with spin one ($m_s = 1, 0, -1$) is obtained

$$
\begin{aligned}
\triangle\phi_\theta &= \theta\xi_3 \oint \vec{\nabla} f^{(1)} \cdot d\vec{r} \\
&= \frac{\theta}{2}\xi_3\varepsilon^{ij} \left(\mu \oint k_j(\partial_i E_2 dx_1 - \partial_i E_1 dx_2) \right. \\
&\left. - 2m_s \oint [(\mu\partial_i E_2)\ \mu(\hat{k}\times\vec{E})_j dx_1 - (\mu\partial_i E_1)\ \mu(\hat{k}\times\vec{E})_j dx_2] \right)
\end{aligned}
\tag{43}
$$

where $m_s = 1, 0, -1$. The first term is a velocity dependent correction and does not have the topological properties of the commutative AC effect and could modify the phase shift. The second term is a correction to the vortex and does not contribute to the line spectrum. Using $m_s = 1/2, -1/2$ the integral in (43) can be mapped to the corrections which have already been obtained for the spin half Aharonov-Casher effect in [17, Eq.(32)]. One may conjecture that (43) is also valid for higher spins. It is interesting to extend these results to higher order terms, however it seems that obtaining an exact result similar to the commutative case is not possible by these methods. For some other interesting relevant papers see [27-34].

References

[1] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73** (2001) 977-1029. hep-th/0106048.

[2] A. Connes, M. R. Douglas and A. Schwarz, JHEP **9802** (1998) 003, hep-th/9808042.

[3] N. Seiberg and E. Witten, JHEP **9909** (1999) 032. hep-th/9908142.

[4] L. Susskind, hep-th/0101029.

[5] M. Chaichian, A. Demichev and P. Presnajder, Nucl. Phys. **B567** 360 (2000), hep-th/9812180.

[6] L. Alvarez-Gaume, S.R. Wadia, Phys. Lett. **B501** 319 (2001), hep-th/0006219.

[7] L. Mezincescu, hep-th/0007046.

[8] V. P.Nair, Phys. Lett. **B505** (2001) 249, hep-th/0008027.

[9] M. Chaichian, M.M. Sheikh-Jabbari and A.Tureanu Phys. Rev. Lett. **86**(2001) 2716, hep-th/0010175.

[10] J. Gamboa, M. Loewe and J. C. Rojas, Phys. Rev. **D64** (2001) 067901, hep-th/0010220.

[11] S. Belluchi , A. Nersessian and C. Sochichiu, Phys. Lett. **B522** (2001) 345, hep-th/0106138.

[12] C. Acatrinei, JHEP **0109** (2001) 007, hep-th/0107078.

[13] J. Gamboa, M. Loewe, F.M. Mendez,J.C. Rojas, Int. J. Mod. Phys. **A17** (2002) 2555-2566, hep-th/0106125.

[14] J. Gamboa, M. Loewe, F. Mendez and J.C.Rojas, Mod. Phys. Lett. **A16** (2001) 2075-2078, hep-th/0104224.

[15] Y. Aharonov and D. Bohm, Phys. Rev. **115** (1958) 485.

[16] M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari and A. Tureanu, Phys. Lett. **B527**, 149-154 (2002), hep-th/0012175.

[17] B. Mirza, M. Zarei, Eur. Phys. J. C. **32**,583-586 (2004), hep-th/0311222.
[18] Y. Aharonov and A. Casher, Phys. Rev. Lett. **53** (1984) 319.
[19] A. Cimmino, G.I. Opat, A.G. Klein, H.Kaiser, S.A. Werner, M.Arif and R. Clothier, Phys. Rev. Lett. **63** (1989) 380.
[20] A.S. Goldhaber, Phys. Rev. Lett. **62** (1989) 482.
[21] T.H. Boyer, Phys. Rev. **A36** (1987) 5083.
[22] Y. Aharonov, P. Pearl and L. Vaidman, Phys. Rev. **A37** (1988) 4052.
[23] Xiao-Gang He and Bruce H.J. McKeller, Phys Lett. **B256** (1991) 250.
[24] J. A. Swansson and B. H. J. McKellar, J. Phys. A: Math Gen. **34** (2001) 1051-1061, quant-ph/0007118.
[25] W. Greiner, Relativistic Quantum Mechanics, Springer-Verlag, 1990.
[26] H. Umezawa, Quantum Field Theory, North-Holland, 1956.
[27] I. Hinchliffe, N. Kersting, Y.L. Ma, Review of the Phenomenology of Noncommutative Geometry,Int.J.Mod.Phys. A19 (2004) 179-204, hep-ph/0205040.
[28] S. A. Alavi, Mod. Phys. Lett. **A20** (2005) 1013.
[29] Li, Kang; Dulat, Sayipjamal, Eur. Phys. J. **C46** (2006) 825-828.
[30] B. Basu, S. Ghosh, S. Dhar, hep-th/0604068.
[31] P. Schupp, J. Trampetic, J. Wess and G. Raffelt, Eur. Phys. J. **C36** (2004) 405-410, hep-ph/0212292.
[32] Ya. I. Azimov and R. M. Ryndin, Phys. Atom. Nucl. 61 (1998) 1932-1936, hep-ph/9710433.
[33] C. Duval and P. A. Horvathy, J. Phys. **A34** (2001) 10097-10108, hep-th/0106089.
[34] M. Haghighat, M. M. Ettefaghi, M. Zeinali, Phys.Rev. **D73** (2006) 013007
[35] L.P. Colatto , A.L.A. Penna , W.C. Santos, Phys.Rev. **D73** (2006) 105007.
[36] E. Passos, L. R. Ribeiro, C. Furtado, J. R. Nascimento, hep-th/0610222.